Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Subtract each row from the one which follows it, beginning with the last but one. Repeat the same operation, stopping at the second row. Keep repeating this operation, leaving out a row each time, until all the rows have been thus omitted; then if D=value of determinant and

$$\triangle^r a_s^2 = \triangle^{r-1} a^2_{s+1} - \triangle^{r-1} a_s^2,$$

we get

$$D = \begin{vmatrix} a_1^2, & a_2^2, & a_3^2, & \dots & a_n^2 \\ \triangle a_1^2, & \triangle a_2^2, & \triangle a_3^2, & \dots & \triangle a_n^2 \\ \triangle^2 a_1^2, & \triangle^2 a_2^2, & \triangle^2 a_3^2, & \dots & \triangle^2 a_n^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \triangle^{n-1} a_1^2, & \triangle^{n-1} a_2^2, & \triangle^{n-1} a_3^2, & \dots & \triangle^{n-1} a_n^2 \end{vmatrix}$$

Repeating the same series of operations on the columns, we get

$$D = \begin{vmatrix} a_1^2, & \triangle a_1^2, & \triangle^2 a_1^2, & \dots & \triangle^{n-1} a_1^2 \\ \triangle a_1^2, & \triangle^2 a_1^2, & \triangle^3 a_1^2, & \dots & \triangle^n a_1^2 \\ \triangle^2 a_1^2, & \triangle^3 a_1^2, & \triangle^4 a_1^2, & \dots & \triangle^{n+1} a_1^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \triangle^{n-1} a_1^2, & \triangle^n a_1^2, & \triangle^{n+1} a_1^2, & \dots & \triangle^{2n-2} a_1^2 \end{vmatrix}$$

If a_r^2 is a function of r of the pth degree in r, whose highest term has a coefficient unity, the quantities a_1^2, a_2^2, a_3^2, ... form an arithmetic series of the pth order.

If $p=n-1$ all the elements below the second diagonal vanish, while all those in it are equal to $(n-1)!$, and $D=(-1)^{n(n-1)/2}\,[(n-1)!]^n$.

If $m<(n-1)$, $D=0$.

These determinants have been called orthosymmetrical.

GEOMETRY.

361. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

ABCD is a quadrilateral. The bisectors of A and C meet in O_1; those of B and D meet in O_2. Find the tangent of the angle between AD and O_1O_2 in terms of sines and cosines of A, D, $A+B$, and $A+D$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $ABCD$ be the quadrilateral. Produce AB, DC, and AD, BC until they intersect in E, F, respectively. Take ADE as the triangle of reference for trilinear coordinates.

Let $\gamma=0$, be the epuation to AD; $\beta=0$, the equation to AB; $\alpha=0$, the equation to DC; $l\alpha+m\beta+n\gamma=0$, the equation to BC.

Also let $P=\sqrt{[l^2+m^2+n^2-2mn\cos A-2nl\cos D-2ml\cos E]}$.

(1) $\alpha-\gamma=0$, bisects angle D.
(2) $l\alpha+m\beta+n\gamma-P\beta=0$, bisects angle B.
(3) $\beta-\gamma=0$, bisects angle A.
(4) $l\alpha+m\beta+n\gamma-P\alpha=0$, bisects angle C.
(1) and (2) intersect in

$$\frac{\alpha_1}{P-m}=\frac{\beta_1}{l+n}=\frac{\gamma_1}{P-m}=\frac{2\triangle}{(a+c)(P-m)+b(n+l)}=O_2.$$

(3) and (4) intersect in

$$\frac{\alpha_2}{n+m}=\frac{\beta_2}{P-l}=\frac{\gamma_2}{P-l}=\frac{2\triangle}{a(n+m)+(b+c)(P-l)}=O_1.$$

Equation to O_1O_2 is

(5) $\alpha(P-l)+\beta(P-m)-\gamma(P+n)=0.$

The angle between (5) and $\gamma=0$ is

$$\tan\phi=\frac{\sin A-\sin D+(l\sin D-m\sin A)/P}{1+\cos A+\cos D+(n-l\cos D-m\cos A)/P}.$$

But angle $(180°-F)=(A+B)=$angle BC makes with AD.
$\therefore \sin(A+B)=(l\sin D-m\sin A)/P$;
$\cos(A+B)=(n-l\cos D-m\cos A)/P$.

$$\therefore \tan\phi=\frac{\sin A-\sin D+\sin(A+B)}{1+\cos A+\cos D+\cos(A+B)}.$$

362. Proposed by V. M. SPUNAR, M. and E. E., 3536 Massachusetts Avenue, N. S., Pittsburg, Pa.

Show that the focus of an ellipse may be regarded as an indefinitely small circle having double contact with the ellipse, the directrix being the chord joining the points of contact.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

A circle with its center at $(x_0, 0)$ any point of the major axis inside the evolute $[x_0<ae^2]$, and having for its radius the length of the normal which meets the axis in that point, is tangent to the ellipse at two points, say $(x_1, \pm y_1)$. From the equation of the normal to $b^2x^2+a^2y^2=a^2b^2$ at (x_1, y_1) we find, since $a^2-b^2=a^2e^2$, $x_0=e^2x_1$; or $x_1=x_0/e^2$. Also the normal length is given by $N^2=(x_1-x_0)^2+y_1^2$, which reduces to $N^2=(1-e^2)\times(a^2-e^2x_1^2)=(1-e^2)(a^2e^2-x_0^2)/e^2$. Thus the circle has the equation